

SHIFTED JACK POLYNOMIALS, BINOMIAL FORMULA, AND APPLICATIONS

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ABSTRACT. In this note we prove an explicit binomial formula for Jack polynomials and discuss some applications of it.

1. Jack polynomials ([M,St]). In this note we use the parameter

$$\theta = 1/\alpha$$

inverse to the standard parameter α for Jack polynomials. Jack symmetric polynomials $P_\lambda(x_1, \dots, x_n; \theta)$ are eigenfunctions of Sekiguchi differential operators

$$(1.1) \quad D(u; \theta) = V(x)^{-1} \det \left[x_i^{n-j} \left(x_i \frac{\partial}{\partial x_i} + (n-j)\theta + u \right) \right]_{1 \leq i, j \leq n},$$

$$D(u; \theta) P_\lambda(x; \theta) = \left(\prod_i (\lambda_i + (n-i)\theta + u) \right) P_\lambda(x; \theta),$$

where $V(x) = \prod_{i < j} (x_i - x_j)$ is the Vandermonde determinant and u is a parameter. The operators $\{D(u; \theta), u \in \mathbb{C}\}$ generate a commutative algebra; denote it by $\mathfrak{D}(n; \theta)$.

We normalize $P_\lambda(x_1, \dots, x_n; \theta)$ so that

$$P_\lambda(x_1, \dots, x_n; \theta) = x_1^{\lambda_1} \dots x_n^{\lambda_n} + \dots,$$

where dots stand for lower monomials in lexicographic order. Then one has

$$(1.2) \quad P_{(\lambda_1+1, \dots, \lambda_n+1)}(x; \theta) = \left(\prod x_i \right) P_\lambda(x; \theta).$$

Using (1.2) one can define Jack rational functions for any integers $\lambda_1 \geq \dots \geq \lambda_n$ and still (1.1), and hence the binomial theorem below, will hold.

Note that the eigenvalues in (1.1) are symmetric in variables $\lambda_i - \theta i$, $i = 1, \dots, n$.

The authors were supported by the Russian Basic Research Foundation (grant 95-01-00814). The first author was supported by the NSF grant DMS 9304580.

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2. Shifted Jack polynomials ([S1, OO, Ok1, KS, Ok3]). Denote by $\Lambda^\theta(n)$ the algebra of polynomials $f(x_1, \dots, x_n)$ that are symmetric in variables $x_i - \theta i$. We call such polynomials *shifted symmetric*. By (1.1) one has the Harish-Chandra isomorphism

$$(2.1) \quad \mathfrak{D}(n; \theta) \rightarrow \Lambda^\theta(n)$$

which takes an operator $D \in \mathfrak{D}(n; \theta)$ to the polynomial $d \in \Lambda^\theta(n)$ such that

$$D P_\lambda(x; \theta) = d(\lambda) P_\lambda(x; \theta).$$

If D is of order k then $\deg d = k$.

Let μ be a partition (also viewed as a diagram). Recall that for a square $s = (i, j) \in \mu$ the numbers

$$\begin{aligned} a(s) &= \mu_i - j, & a'(s) &= j - 1, \\ l(s) &= \mu'_j - i, & l'(s) &= i - 1, \end{aligned}$$

are called arm-length, arm-colength, leg-length, and leg-colength respectively. Set

$$H(\mu) = \prod_{s \in \mu} (a(s) + \theta l(s) + 1).$$

According to a general result of S. Sahi [S1] there exists the unique polynomial $P_\mu^*(x; \theta)$ in $\Lambda^\theta(n)$ such that $\deg P_\mu^* \leq |\mu|$ and

$$P_\mu^*(\lambda; \theta) = \begin{cases} H(\mu), & \lambda = \mu, \\ 0, & |\mu| \leq |\lambda|, \mu \neq \lambda \end{cases}$$

Here we assume μ and λ have length $\leq n$. It is clear that the uniqueness of $P_\mu^*(x, \theta)$ follows easily from its existence.

The polynomials $P_\mu^*(x; 1)$ were studied in [OO] and [Ok1]. They are closely related to Schur polynomials and are called *shifted Schur functions*. In particular, the Schur function s_μ is the highest degree term of $P_\mu^*(x; 1)$. Shifted Schur functions have a very rich combinatorics and numerous applications (such as Capelli-type identities or asymptotic character theory, see [OO, Ok1, N, Ok2] and references therein).

For general θ , F. Knop and S. Sahi proved (see [KS], Theorem 5.2) that

$$(2.2) \quad P_\mu^*(\lambda; \theta) = 0, \quad \text{unless } \mu \subset \lambda,$$

and that again ([KS], Corollary 4.7)

$$(2.3) \quad P_\mu^*(x; \theta) = P_\mu(x; \theta) + \text{lower degree terms}.$$

Their proof was based on difference equations for the polynomial $P_\mu^*(x; \theta)$. For $\theta = 1$ these properties follow immediately from explicit formulas for $P_\mu^*(x; 1)$.

Explicit formulas for the polynomials $P_\mu^*(x; \theta)$, which generalize explicit formulas for shifted Schur functions, were found in [Ok2] (see, for example, (2.4) below). In

particular, they provide an effective proof of the existence of $P_\mu^*(x; \theta)$ and a different proof of (2.2) and (2.3).

We call these polynomials *shifted Jack polynomials*. They are a degenerate case of shifted Macdonald polynomials [Kn,S2,Ok3].

It easily follows from the definition that $P_\mu^*(x; \theta)$ are stable, that is

$$P_\mu^*(x_1, \dots, x_n, 0; \theta) = P_\mu^*(x_1, \dots, x_n; \theta)$$

provided μ has at most n parts. Hence, one can consider shifted Jack polynomials in infinitely many variables.

The following *combinatorial formula* for shifted Jack polynomials was proved in [Ok3]. Let us call a tableau T on μ a *reverse tableau* if its entries strictly decrease down the columns and weakly decrease in the rows. By $T(s)$ denote the entry in the square $s \in \mu$. We have

$$(2.4) \quad P_\mu^*(x; \theta) = \sum_T \psi_T(\theta) \prod_{s \in \mu} (x_{T(s)} - a'(s) + \theta l'(s)) ,$$

where the sum is over all reverse tableaux on μ with entries in $\{1, 2, \dots\}$ and $\psi_T(\theta)$ is the same θ -weight of a tableau that enters the combinatorial formula for ordinary Jack polynomials (see [St] or [M],(VI.10.12))

$$(2.5) \quad P_\mu(x; \theta) = \sum_T \psi_T(\theta) \prod_{s \in \mu} x_{T(s)} .$$

The coefficients $\psi_T(\theta)$ are rational functions of θ .

3. Binomial formula. Given a partition μ and a number t set

$$\begin{aligned} H'(\mu) &= \prod_{s \in \mu} (a(s) + \theta l(s) + \theta) , \\ (t)_\mu &= \prod_{s \in \mu} (t + a'(s) - \theta l'(s)) . \end{aligned}$$

If $\mu = (m)$ then $(t)_\mu$ is the standard shifted factorial. We have (see [M], VI.10.20)

$$(3.1) \quad P_\mu(1, \dots, 1; \theta) = (n\theta)_\mu / H'(\mu) .$$

Set (see [M], VI.10.16)

$$Q_\mu(x; \theta) = \frac{H'(\mu)}{H(\mu)} P_\mu(x; \theta) .$$

The main result of this note is the following

Theorem.

$$(3.2) \quad \frac{P_\lambda(1 + x_1, \dots, 1 + x_n; \theta)}{P_\lambda(1, \dots, 1; \theta)} = \sum_\mu \frac{P_\mu^*(\lambda_1, \dots, \lambda_n; \theta) Q_\mu(x_1, \dots, x_n; \theta)}{(n\theta)_\mu} .$$

Note that by virtue of (2.2) the summation in the binomial formula (3.2) is only over such μ that

Let \mathcal{O}_1 be the local ring of symmetric rational functions regular at the point

$$(3.3) \quad \bar{1} = (1, \dots, 1)$$

and let $\mathfrak{m}_1 \subset \mathcal{O}_1$ be its maximal ideal. We call functionals

$$\psi \in (\mathcal{O}_1/\mathfrak{m}_1^k)^* \subset \mathcal{O}_1^*$$

symmetric distributions of order $\leq k$ supported at the point $\bar{1}$ defined in (3.3). Denote by \mathcal{S}_1 the space of all symmetric distributions supported at $\bar{1}$. Looking at the highest derivatives one easily proves that the map

$$(3.4) \quad \mathfrak{D}(n; \theta) \rightarrow \mathcal{S}_1$$

which takes an operator D to the distribution ψ_D

$$\psi_D(f) = (Df)(1, \dots, 1)$$

is an order preserving isomorphism of linear spaces.

Proof of the theorem. Consider the Taylor expansion of $P_\lambda(x; \theta)$ about the point $\bar{1}$. Since Jack polynomials form a linear basis in symmetric polynomials this expansion has the form

$$(3.5) \quad P_\lambda(1 + x_1, \dots, 1 + x_n; \theta) = \sum_{\mu} \psi_{\mu}(P_{\lambda}) P_{\mu}(x; \theta), \quad \psi_{\mu} \in \mathcal{S}_1.$$

Note that ψ_{μ} is of order $\leq |\mu|$. By virtue of the isomorphisms (2.1) and (3.4) we have

$$(3.6) \quad P_\lambda(1 + x_1, \dots, 1 + x_n; \theta) = P_\lambda(1, \dots, 1) \sum_{\mu} f_{\mu}(\lambda) P_{\mu}(x; \theta), \quad f_{\mu} \in \Lambda^{\theta}(n),$$

where $\deg f_{\mu} \leq |\mu|$. On the other hand

$$(3.7) \quad P_\lambda(1 + x_1, \dots, 1 + x_n; \theta) = P_\lambda(x_1, \dots, x_n; \theta) + \text{lower degree terms}.$$

From (3.6) and (3.7) we obtain

$$f_{\mu}(\lambda) = \begin{cases} 0, & |\lambda| \leq |\mu|, \lambda \neq \mu, \\ 1/P_{\mu}(1, \dots, 1), & \lambda = \mu. \end{cases}$$

Since f_{μ} is a polynomial of degree $\leq |\mu|$ it is completely determined by its values at the points λ , $|\lambda| \leq |\mu|$. Therefore, f_{μ} is proportional to P_{μ}^* and by (3.1)

$$f_{\mu}(x) = \frac{H'(\mu)}{(n\theta)_{\mu} H(\mu)} P_{\mu}^*(x; \theta),$$

which concludes the proof. \square

Remark. The coefficients $\binom{\lambda}{\mu}_\theta$ in the expansion

$$\frac{P_\lambda(1+x_1, \dots, 1+x_n; \theta)}{P_\lambda(1, \dots, 1; \theta)} = \sum_{\mu} \binom{\lambda}{\mu}_\theta \frac{P_\mu(x_1, \dots, x_n; \theta)}{P_\mu(1, \dots, 1; \theta)}.$$

are called the *generalized binomial coefficients*. They were studied by C. Bingham [Bi] in the special case $\theta = 1/2$, using group-theoretical methods, and by M. Lassalle [La] for general θ . Then a proof of the main results announced in [La] was proposed by J. Kaneko (see [K], section 5). Our theorem says that this binomial coefficient

$$(3.8) \quad \binom{\lambda}{\mu}_\theta = \frac{P_\mu^*(\lambda; \theta)}{H(\mu)}$$

is a shifted symmetric polynomial in λ which by (2.4) and (2.5) is just as complex as the Jack polynomial $P_\mu(x; \theta)$ itself. The main results of Lassalle can be easily deduced from (3.8). For example, it is clear that (3.8) does not depend on n and vanishes unless $\mu \subset \lambda$. The recurrence relation (see [La], théorèmes 2, 3, 5 and corollaire in section 6) is equivalent to the formula (5.2) below.

Note also that the coefficients (3.8) admit a simple formula when $\theta = 1$ (see A. Lascoux [Lasc] and Example 10 in [M], section I.3) or when $\lambda_1 = \dots = \lambda_n$ (see J. Faraut and A. Korányi [FK], Prop. XII.1.3 (ii)).

Binomial formulas for characters of classical groups are discussed in [OO2]³.

4. Bessel functions ([D,Op,J]). For a real vector $l = (l_1, \dots, l_n)$ denote by

$$[l] = ([l_1], \dots, [l_n])$$

its integral part. Suppose that $l_1 \geq \dots \geq l_n$. By definition, put

$$(4.1) \quad F(l, x; \theta) = \lim_{\kappa \rightarrow \infty} \frac{P_{[\kappa l]}(1+x_1/\kappa, \dots, 1+x_n/\kappa; \theta)}{P_{[\kappa l]}(1, \dots, 1; \theta)}.$$

From (3.2) and (2.3) we have

Proposition.

$$(4.2) \quad F(l, x; \theta) = \sum_{\mu} \frac{P_\mu(l_1, \dots, l_n; \theta) Q_\mu(x_1, \dots, x_n; \theta)}{(n\theta)_\mu}.$$

Proof. We have (one easily checks absolute convergence of all series)

$$\begin{aligned} F(l, x; \theta) &= \lim_{\kappa \rightarrow \infty} \frac{P_{[\kappa l]}(1+x_1/\kappa, \dots, 1+x_n/\kappa; \theta)}{P_{[\kappa l]}(1, \dots, 1; \theta)} \\ &= \sum_{\mu} \lim_{\kappa \rightarrow \infty} \frac{P_\mu^*([\kappa l_1], \dots, [\kappa l_n]; \theta) Q_\mu(x_1/\kappa, \dots, x_n/\kappa; \theta)}{(n\theta)_\mu} \\ &= \sum_{\mu} \lim_{\kappa \rightarrow \infty} \frac{\kappa^{-|\mu|} P_\mu^*([\kappa l_1], \dots, [\kappa l_n]; \theta) Q_\mu(x_1, \dots, x_n; \theta)}{(n\theta)_\mu} \\ &= \sum_{\mu} \frac{P_\mu(l_1, \dots, l_n; \theta) Q_\mu(x_1, \dots, x_n; \theta)}{(n\theta)_\mu}, \end{aligned}$$

³Recently the first author found a binomial formula for Macdonald polynomials (both shifted and ordinary) which for Kronecker polynomials coincides with [OO2, 5].

where the second equality is based on the binomial formula and the last equality follows from (2.3) \square .

We call $F(l, x; \theta)$ the *Bessel functions*. They are in the same relation to Jack polynomials as ordinary Bessel functions to Jacobi polynomials. They are eigenfunctions of the corresponding degeneration of Sekiguchi operators. The formula (4.2) makes obvious the following known symmetry

$$F(l, x; \theta) = F(x, l; \theta).$$

Let $H(n, \mathbb{R})$ denote the space of real symmetric matrices of order n , let $X, Y \in H(n, \mathbb{R})$, and let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be the spectra of X and Y . The compact orthogonal group $O(n)$ acts on the space $H(n, \mathbb{R})$ by conjugations. One can prove that

$$(4.3) \quad F(y, x; 1/2) = \int_{O(n)} \exp(\text{tr}(Y u X u^{-1})) du,$$

where du is the normalized Haar measure on $O(n)$. (Idea of proof: the normalized polynomials

$$P_\lambda(z_1, \dots, z_n; 1/2) / P_\lambda(1, \dots, 1; 1/2)$$

are the spherical functions on the compact symmetric space $U(n)/O(n)$ while the integrals (4.3) are essentially the spherical functions on the associated Euclidean type symmetric space $O(n) \ltimes H(n, \mathbb{R})$. It is well-known (see [DR]) that the spherical functions of Euclidean type can be obtained from the spherical functions of compact type by the limit transition (4.1).)

A similar interpretation of the functions $F(y, x; \theta)$ also exists for $\theta = 1$ and $\theta = 2$. Then one has to consider orbits in the spaces $H(n, \mathbb{C})$ and $H(n, \mathbb{H})$ of complex and quaternionic Hermitian matrices, respectively. An application of the expansion (4.2) for $\theta = 1$ is given in [OV]; similar results also hold for $\theta = 1/2$ and $\theta = 2$.

5. Formula for θ -dimension of a skew diagram λ/μ . Define the θ -dimension $\theta\text{-dim } \lambda/\mu$ of a skew diagram λ/μ as the following coefficient

$$(5.1) \quad \left(\sum x_i \right)^k P_\mu(x; \theta) = \sum_{|\lambda|=|\mu|+k} \theta\text{-dim } \lambda/\mu P_\lambda(x; \theta), \quad k = 1, 2, \dots$$

If $\theta = 1$ then $\theta\text{-dim } \lambda/\mu$ equals the number of the standard tableaux on λ/μ ; for general θ each tableau T is counted with a certain weight $\psi'_T(\theta)$ given in [M], section VI.6. In particular, if $\mu = \emptyset$ then (see (5.6) below for a proof using shifted Jack polynomials)

$$\theta\text{-dim } \lambda = |\lambda|! / H(\lambda).$$

We have

Proposition.

$$(5.2) \quad \frac{\theta\text{-dim } \lambda/\mu}{\theta\text{-dim } \lambda} = \frac{P_\mu^*(\lambda; \theta)}{|\lambda|(|\lambda| - 1) \cdots (|\lambda| - |\mu| + 1)}.$$

This proposition can be deduced from (3.8) and [La, K] (see the remark above), but it is easier to give a direct proof, which uses the very same argument as in [OO], section 9.

First one proves that

Lemma.

$$(5.3) \quad \left(\sum x_i - |\mu| \right) P_\mu^*(x; \theta) = \sum_{|\lambda|=|\mu|+1} \theta \text{-dim } \lambda / \mu P_\lambda^*(x; \theta).$$

Proof. Let f be the difference of the LHS and RHS in (5.3). By (2.3) and (5.1) we have $\deg f \leq |\mu|$. One easily checks that $f(\nu) = 0$ for all $|\nu| \leq |\mu|$. It follows that $f = 0$. \square

Applying this lemma k times, where $k = |\lambda| - |\mu|$, we obtain

$$(5.4) \quad \left(\sum x_i - |\mu| \right) \cdots \left(\sum x_i - |\mu| - k + 1 \right) P_\mu^*(x; \theta) = \sum_{|\nu|=|\mu|+k} \theta \text{-dim } \nu / \mu P_\nu^*(x; \theta).$$

Evaluation of (5.4) at $x = \lambda$ gives

$$(5.5) \quad (|\lambda| - |\mu|)! P_\mu^*(\lambda) = \theta \text{-dim } \lambda / \mu P_\lambda^*(\lambda).$$

In particular, for $\mu = \emptyset$ one has

$$(5.6) \quad |\lambda|! = \theta \text{-dim } \lambda P_\lambda^*(\lambda).$$

Dividing (5.5) by (5.6) we obtain (5.2). \square

6. Integral representation of Jack polynomials. Write $\nu \prec \lambda$ if

$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_n.$$

For any number r set $(t)_r = \Gamma(t+r)/\Gamma(t)$. We have (see [M], VI.7.13')

$$(6.1) \quad P_\lambda(x_1, \dots, x_n; \theta) = \sum_{\nu \prec \lambda} \psi_{\lambda/\nu} P_\nu(x_1, \dots, x_{n-1}) x_n^{|\lambda/\nu|},$$

where

$$(6.2) \quad \psi_{\lambda/\nu} = \prod_{i \leq j \leq n-1} \frac{(\nu_i - \lambda_{j+1} + \theta(j-i) + 1)_{\theta-1} (\lambda_i - \nu_j + \theta(j-i) + 1)_{\theta-1}}{(\lambda_i - \lambda_{j+1} + \theta(j-i) + 1)_{\theta-1} (\nu_i - \nu_j + \theta(j-i) + 1)_{\theta-1}}.$$

Suppose μ has less than n parts. The formulas (3.2) and (6.1) together imply

$$(6.3) \quad P_\mu^*(\lambda) = \frac{(n\theta)_\mu}{((n-1)\theta)_\mu} \sum_{\nu \prec \lambda} \psi_{\lambda/\nu} \frac{P_\nu(1, \dots, 1)}{P_\lambda(1, \dots, 1)} P_\mu^*(\nu)$$

where 1 is repeated n times in the denominator and $(n-1)$ times in the numerator. Note that after substitution of (6.2) and of (3.1) rewritten as

$$(6.4) \quad P_\lambda(1, \dots, 1) = \prod_{i < j \leq n} (\lambda_i - \lambda_j + \theta(j-i))_\theta \prod_{i \leq n} \Gamma(\theta)/\Gamma(\theta i)$$

the formula (6.3) becomes the formula (7.16) from [Ok3]. We replace λ and μ in (6.3) by their multiples $\kappa\lambda$ and $\kappa\mu$ and let $\kappa \rightarrow \infty$. By (2.3)

$$P_\mu^*(\kappa\lambda)/\kappa^{|\mu|} \rightarrow P_\mu(\lambda), \quad \kappa \rightarrow \infty.$$

Introduce the following product of beta-functions

$$C(\mu, n) = \prod_{i \leq n-1} B(\mu_i + (n-i)\theta, \theta).$$

Set

$$\Pi(\lambda, \nu; \theta) = \prod_{i \leq j} (\lambda_i - \nu_j)^{\theta-1} \prod_{i > j} (\nu_j - \lambda_i)^{\theta-1}.$$

Using the well known relation $(t)_\theta/t^\theta \rightarrow 1$, $t \rightarrow \infty$, one obtains from (6.3) the following

Proposition. *If μ has $< n$ parts then*

$$P_\mu(\lambda) = \frac{1}{C(\mu, n)} \frac{1}{V(\lambda)^{2\theta-1}} \int_{\lambda_2}^{\lambda_1} d\nu_1 \dots \int_{\lambda_n}^{\lambda_{n-1}} d\nu_{n-1} P_\mu(\nu) V(\nu) \Pi(\lambda, \nu; \theta).$$

Here $\lambda_1 \geq \dots \geq \lambda_n$ are arbitrary real (this assumption is not essential since P_μ is symmetric). If $\theta = 1, 2, \dots$ then the integrand is holomorphic and λ can be arbitrary complex. Iteration of this proposition together with (1.2) gives an integral representation for all Jack polynomials.

This proposition was first obtained by G. Olshanski (unpublished) for special values $\theta = 1/2, 1, 2$ by using the group theoretic interpretation of the corresponding Jack polynomials. Then A. Okounkov [Ok3] found a general method which works for any θ and even for Macdonald polynomials. The proof presented above differs from that from [Ok3].

7. Other applications. The binomial theorem itself and the formula (5.2) result in an θ -analog of the Vershik-Kerov theorems [VK1, VK2] about characters of $U(\infty)$ and $S(\infty)$. These results will be discussed elsewhere.

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